TOPIC- “LIMIT”

Objectives

- Know what left limits, right limits, and limits are
- Know how to compute simple limits
- Know what it means for a function to be continuous
- Know how $\sin(x)$ and $\cos(x)$ behave as $x \to 0$.

What is a limit?

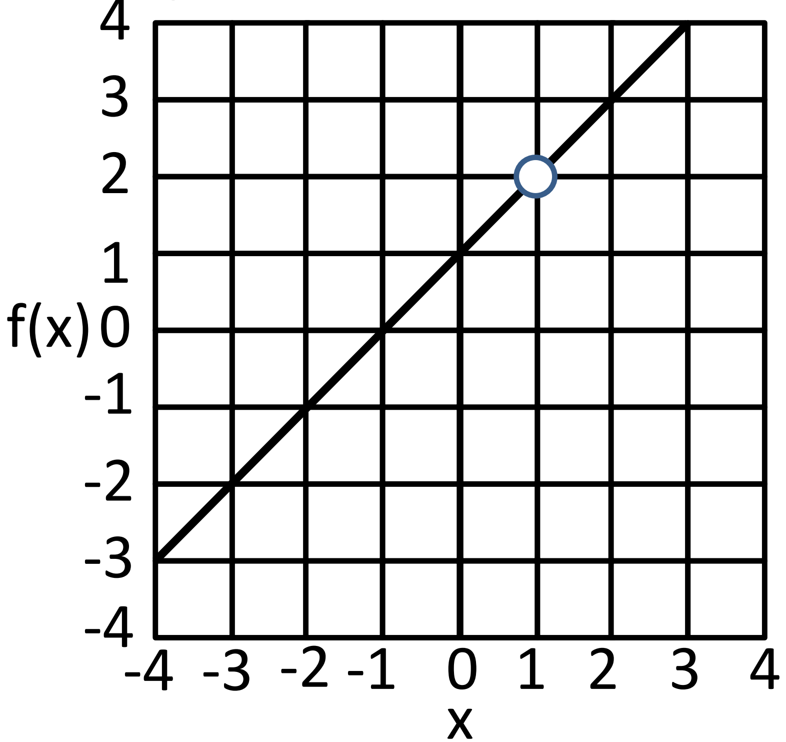
- A limit is what happens when you get closer and closer to a point without actually reaching it.
- Example: If $f(x) = 2x$ then as $x \to 1$, $f(x) \to 2$.
- We write this as $\lim_{x \to 1} f(x) = 2$.

x	0	.9	.99	.999	.9999
f(x)	0	1.8	1.98	1.998	1.9998

Why are limits useful?

- Many functions are not defined at a point but are well-behaved nearby.
- Example: If $f(x) = \frac{x^2-1}{x-1}$ then $f(1)$ is undefined. However, as $x \to 1$, $f(x) \to 2$, so $\lim_{x\to 1} f(x) = 2$.

x	0	.9	.99	.999	.9999
f(x)	0	1.9	1.99	1.999	1.9999



Left Limits and Right Limits

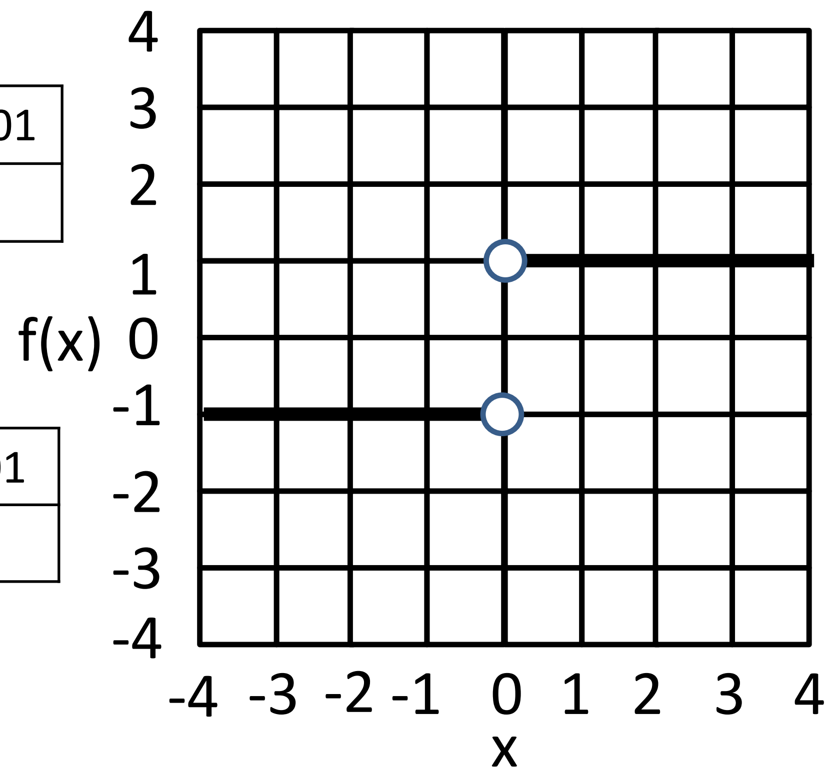
Consider $f(x) = \frac{x}{|x|}$. $f(0)$ is undefined.

As $x \to 0^-, f(x) = -1$

x	-1	-.1	-.01	-.001	-.0001
f(x)	-1	-1	-1	-1	-1

As $x \to 0^+, f(x) = 1$

x	1	.1	.01	.001	.0001
f(x)	1	1	1	1	1



We write this as $\lim_{x \to 0^-} f(x) = -1, \lim_{x \to 0^+} f(x) = 1$

Limit Definition Summary

- We say that $\lim_{x \to a^-} f(x) = L$ if $f(x) \to L$ as $x \to a^-$
- We say that $\lim_{x \to a^+} f(x) = L$ if $f(x) \to L$ as $x \to a^+$
- If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$ (i.e. it doesn't matter which side x approaches a from) then we say that $\lim_{x \to a} f(x) = L$

Non existence of Limits

- Limits can fail to exist in several ways
- 1. $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ may not exist.
- Example: $\sin\left(\frac{1}{x}\right)$ oscillates rapidly between 0 and 1 as $x \to 0^+$ (or 0^-). Thus, $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ DNE (does not exist)
- Example: $\frac{1}{x}$ gets larger and larger as $x \to 0^+$. We write this as $\lim_{x \to 0^+} \frac{1}{x} = \infty$
- 2. $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ may both exist but have different values. Ex: $f(x) = \frac{x}{|x|}$ near $x = 0$

Computing Limits

- To compute $\lim_{x \to a} f(x)$:
- If nothing special happens at $x = a$, just compute $f(a)$. Example: $\lim_{x \to 2}(3x - 1) = 5$
- If plugging in $x = a$ gives $\frac{0}{0}$, factors can often be cancelled when $x \neq a$.

  Example:

$$\lim_{x \to 2}\left(\frac{x^2 - 4}{x - 2}\right) = \lim_{x \to 2}\left(\frac{(x-2)(x+2)}{x - 2}\right) = \lim_{x \to 2}(x + 2) = 4$$

Computing Limits Continued

- Useful trick: $a - b = (a - b) \cdot \frac{a+b}{a+b} = \frac{a^2 - b^2}{a+b}$
- Example: What is $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$?

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$$

$$= \lim_{x \to 0} \frac{x}{x(\sqrt{x+1}+1)} = \lim_{x \to 0} \frac{1}{(\sqrt{x+1}+1)} = \frac{1}{2}$$

Limits at Infinity

- We can also consider what happens when $x \to \infty$ or $x \to -\infty$. Example: Consider $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$. As $\mathrm{x} \to \infty$ (or $-\infty$), $f(x) \to 1$. We write this as $\lim_{x \to \infty} \frac{x-1}{x} = 1$

Computing Limits at $\pm\infty$

- General strategy : figure out the largest terms and ignore everything else
- Example: If $f(x) = \frac{3x^2 - x}{4x^2 + 2x - 5}$, as $x \to \infty$ only the $3x^2$ in the numerator and the $4x^2$ will really matter, so $\lim_{x \to \infty} f(x) = \frac{3}{4}$

Growth Rates as $x \to \infty$

- As $x \to \infty$, $1 \ll \ln x \ll x^n \ll c^x$ as long as $n > 0$ and $c > 1$
- Examples:
- $\lim_{x\to\infty} \frac{lnx}{\sqrt{x}} = 0$
- $\lim_{x\to\infty} \frac{2^x}{5x^{100}} = \infty$

Limit Laws

- If $\lim_{x \to a} f(x) = \mathrm{L}$ and $\lim_{x \to a} g(x) = M$ then:
- $\lim_{x \to a}(f(x) + g(x)) = \mathrm{L} + \mathrm{M}$
- $\lim_{x \to a}(f(x) - g(x)) = \mathrm{L} - \mathrm{M}$
- $\lim_{x \to a}(f(x)g(x)) = \mathrm{LM}$
- $\lim_{x \to a}(\frac{f(x)}{g(x)}) = \frac{L}{M}$ (if $M \neq 0$)
- Etc.

THANK YOU FOR WATCHING

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