B

A

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TANGENTS and NORMAL

Let $y = f(x)$ be a continuous curve and let $P(x_1, y_1)$ be a point on it then the slope PT at $P(x_1, y_1)$ is given by

$$\left(\frac{dy}{dx}\right) \text{at } (x_1, y_1) \qquad ....(i)$$

and (i) is equal to $\tan\theta$

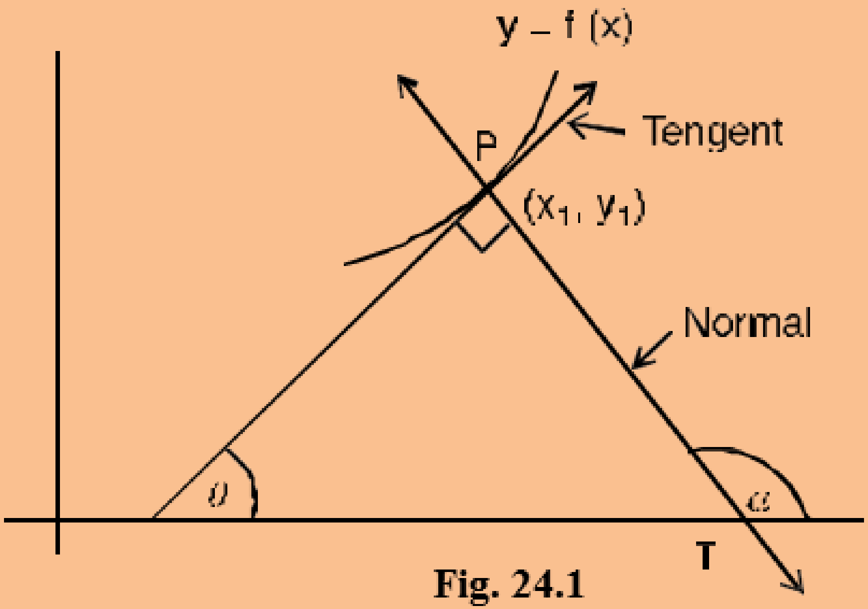

Fig. 24.1

We know that a normal to a curve is a line perpendicular to the tangent at the point of contact

We know that $\alpha = \frac{\pi}{2} + \theta$ (From Fig. 24.1)

$$\Rightarrow \quad \tan\alpha = \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$= -\frac{1}{\tan\theta}$$

$\therefore$ Slope of normal $= -\frac{1}{m} = \frac{-1}{\left(\frac{dy}{dx}\right)}$ at (x_1, y_1) or $-\left(\frac{dx}{dy}\right)$ at (x_1, y_1)

Note

1. The tangent to a curve at any point will be parallel to x-axis if $\theta = 0$, i.e, the derivative at the point will be zero.

i.e. $$\left(\frac{dy}{dx}\right) \text{at } (x_1, y_1) = 0$$

2. The tangent at a point to the curve $y = f(x)$ will be parallel to y-axis if $\frac{dy}{dx} = 0$ at that point.

Let us consider some examples :

Example 24.1 Find the slope of tangent and normal to the curve

$$x^2 + x^3 + 3xy + y^2 = 5 \text{ at } (1,1)$$

Solution : The equation of the curve is

$$x^2 + x^3 + 3xy + y^2 = 5$$

Differentialing (i),w.r.t. x, we get

$$2x + 3x^2 + 3\left[x\frac{dy}{dx} + y.1\right] + 2y\frac{dy}{dx} = 0$$

Substituting $x = 1, \quad y = 1$, in (ii), we get

$$2\times1 + 3\times1 + 3\left[\frac{dy}{dx} + 1\right] + 2\frac{dy}{dx} = 0$$

or $\qquad 5\frac{dy}{dx} = -8 \Rightarrow \qquad \frac{dy}{dx} = -\frac{8}{5}$

$\therefore$ The slope of tangent to the curve at (1, 1) is $-\frac{8}{5}$

$\therefore$ The slope of normal to the curve at (1, 1) is $\frac{5}{8}$

Example 24.2 Show that the tangents to the curve $y = \frac{1}{6}\left[3x^4 + 2x^3 - 3x\right]$ at the points $x = \pm 3$ are parallel.

Solution : The equation of the curve is $y = \dfrac{3x^5 + 2x^3 - 3x}{6}$ .....(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\left(15x^4 + 6x^2 - 3\right)}{6}$$

$$\left(\frac{dy}{dx}\right)_{x=3} \text{at} = \frac{\left[15(3)^4 + 6(3)^2 - 3\right]}{6}$$

$$= \frac{1}{6}[15 \times 9 \times 9 + 54 - 3]$$

$$= \frac{3}{6}[405 + 17] = 211$$

$$\left(\frac{dy}{dx}\right) \text{at} x = -3 = \frac{1}{6}\left[15(-3)^4 + 6(-3)^2 - 3\right] = 211$$

$\therefore$ The tangents to the curve at $x = \pm 3$ are parallel as the slopes at $x = \pm 3$ are equal.

24.2 EQUATIONS OF TANGENT AND NORMAL TO A CURVE

We know that the equation of a line passing through a point (x_1, y_1) and with slope m is

$$y - y_1 = m\,(x - x_1)$$

As discussed in the section before, the slope of tangent to the curve $y = f(x)$ at (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)$ at (x_1, y_1) and that of normal is $\left(-\frac{dx}{dy}\right)$ at (x_1, y_1)

$\therefore$ Equation of tangent to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} [x - x_1]$$

And, the equation of normal to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1, y_1)} [x - x_1]$$

Note

(i) The equation of tangent to a curve is parallel to x-axis if $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$. In that case the equation of tangent is $y = y_1$.

(ii) In case $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} \to$, the tangent at (x_1, y_1) is parallel to $y-$axis and its equation is $x = x_1$

Let us take some examples and illustrate

QUESTION:- Find the points on the curve $\frac{x^2}{9} - \frac{y^2}{16} = 1$ at which the tangents are parallel to x-axis.

Solution : The equation of the curve is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \qquad \text{....(i)}$$

Differentiating (i) w.r.t. x we get

$$\frac{2x}{9} - \frac{2y}{16} \cdot \frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = \frac{16x}{9y}$$

For tangent to be parallel to x-axis, $\frac{dy}{dx} = 0$

$$\Rightarrow \qquad \frac{16x}{9y} = 0 \qquad \Rightarrow \qquad x = 0$$

Putting x = 0 in (i), we get $y^2 = -16\ y = \pm 4i$

This implies that there are no real points at which the tangent to $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is parallel to x-axis.

QUESTION:- Find the equation of all lines having slope – 4 that are tangents to the curve

$$y = \frac{1}{x-1}.$$

Solution : $$y = \frac{1}{x-1} \qquad \text{.....(i)}$$

$\therefore$ $$\frac{dy}{dx} = -\frac{1}{(x-1)^2}$$

It is given equal to -4

$\therefore$ $$\frac{-1}{(x-1)^2} = -4$$

$\Rightarrow$ $$(x-1)^2 = \frac{1}{4}$$

$\Rightarrow$ $$x = 1 \pm \frac{1}{2} \Rightarrow x = \frac{3}{2}, \frac{1}{2}$$

Substituting $x = \frac{1}{2}$ in (i), we get

$$y = \frac{1}{\frac{1}{2} - 1} = \frac{1}{-\frac{1}{2}} = -2$$

$$x = \frac{3}{2}, \quad y = 2$$

$\therefore$ The points are $\left(\frac{3}{2}, 2\right), \left(\frac{1}{2}, -2\right)$

$\therefore$ The equations of tangents are

(a) $y - 2 = -4\left(x - \frac{3}{2}\right)$

$\Rightarrow$ $y - 2 = -4x + 6$ or $4x + y = 8$

(b) $y + 2 = -4\left(x - \frac{1}{2}\right)$

$\Rightarrow$ $y + 2 = -4x + 2$ or $4x + y = 0$

END OF THE PRESENTATION